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Further inequalities involving the Khatri–Rao product

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This paper is dedicated to George P.H. Styan on the occasion of his 70th birthday.

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ABSTRACT

Styan [G.P.H. Styan, Hadamard products and multivariate statistical analysis, Linear Algebra Appl. 6 (1973) 217–240] established an inequality involving the Hadamard product using statistical reasoning in the context of multivariate analysis. In this paper, the inequality is extended to involve the Khatri–Rao product in the non-negative definite matrix case and in the non-singular Hermitian matrix case. The equality conditions for these extensions are given. Also established are counterpart inequalities in the positive definite matrix case.

1. Introduction

Let $C^{m\times n}$ denote the set of $m \times n$ complex matrices and $H(m)$ the set of $m \times m$ Hermitian matrices. For $A \in C^{m\times n}$, let A^H be the conjugate transpose matrix, A^+ the Moore–Penrose inverse of A, and $A^0 = AA^+$ the orthogonal projector on the column space of A. Let $A(\alpha, \beta)$ denote the submatrix of $A \in C^{m\times n}$ with the rows given by α taken from $\langle m\rangle = \{1, 2, \ldots, m\}$ and columns by β taken from $\langle n\rangle = \{1, 2, \ldots, n\}$, and in particular write $A(\alpha, \alpha) = A(\alpha)$. We write $A \geqslant 0$ (or $A > 0$) if A is a Hermitian non-negative (or positive) definite matrix. Let $A \geqslant B$ hold in the Löwner sense, i.e. $A - B \geqslant 0$, $A^{1/2}$ denote the non-negative definite square root of $A \geqslant 0$, $R > 0$ denote a correlation matrix (i.e. its diagonal elements are equal to 1 and all off-diagonal elements are between +1 and -1) and I denote the identity matrix.

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Let $F \in C^{m\times m}$ and $G \in C^{p\times p}$ be multiply partitioned as $F = (F_{ij})$ and $G = (G_{kl})$, where F_{ij} is of size $m_i \times m_j$ and G_{kl} is of size $p_k \times p_l$ ($\sum m_i = m$ and $\sum p_k = p$). Let $F * G = (F_{ij} \otimes G_{ij})$ denote the Khatri–Rao product of F and G, and $F \bowtie G = (F_{ij} \otimes G_{kl})$ denote the Tracy-Singh product, where $\otimes$ denotes the Kronecker product, as in $T \otimes S = (t_{ij}S)$ for any matrices $T = (t_{ij})$ and $S = (s_{ij})$; see [3,13]. The two products are extensions of the Hadamard and Kronecker products respectively, and play a very important role in statistics, engineering and several other areas. See also e.g. [4,15,9,8] for results on the matrix partition and products.

Styan [10, Theorem 4.1, Corollaries 4.2 and 4.3] presented companion inequalities involving the Hadamard product using statistical reasoning in the context of multivariate analysis, for a positive definite correlation matrix R and a Hermitian positive definite matrix A, respectively:

$$2(R \odot R)^{-1} \leqslant R^{-1} \odot R + I, \tag{1}$$
$$2(A \odot I)(A^{-1} \odot A + I)^{-1}(A \odot I) \leqslant A \odot A, \tag{2}$$

where $\odot$ denotes the Hadamard product, as in $T \odot S = (t_{ij}s_{ij})$ for any matrices T and S of the same size. Also, we can write $T \odot S + Y$ instead of $(T \odot S) + Y$ for T, S and Y of the same size (i.e., product precedes sum).

Styan [10] noted that “a matrix-theoretic proof of theorem 4.1 would be of interest”. Subsequent research proceeded along a matrix-theoretic line. Ando [1] gave (3) extending (2), while [14,16] gave (4) for $A, B > 0$:

$$(A \odot I + B \odot I)(A^{-1} \odot B + B^{-1} \odot A + 2I)^{-1}(A \odot I + B \odot I) \leqslant A \odot B, \tag{3}$$
$$(A \odot I + B \odot I)(A \odot B)^{-1}(A \odot I + B \odot I) \leqslant A^{-1} \odot B + B^{-1} \odot A + 2I. \tag{4}$$

Liu [5] proved, in the non-negative definite matrix case ($A, B \geqslant 0$),

$$\begin{aligned}&(A \odot B^0 + B \odot A^0)(A \odot B)^{+}(A \odot B^0 + B \odot A^0)\\ &\quad \leqslant A^{+} \odot B + B^{+} \odot A + 2A^0 \odot B^0.\end{aligned} \tag{5}$$

Liu [6] extended (5) by using the Khatri–Rao product to replace the Hadamard product, for $A, B \geqslant 0$:

$$(A * B^0 + B * A^0)(A * B)^{+}(A * B^0 + B * A^0) \leqslant A^{+} * B + B^{+} * A + 2A^0 * B^0. \tag{6}$$

For positive definite matrices, actually (3) and (4) are identical, and therefore they may be called companion inequalities to each other in this paper. However, for (5) and (6) there is no companion inequality established. For matrices which are only non-singular Hermitian (and are not positive definite), we can provide examples that (4) holds for some but not all of those matrices (see Section 4 for a numerical example), and so do (3), (5) and (6). Furthermore, such inequalities can find their useful applications in statistics and other areas as indicated by e.g. [10,1,8]. So, in this paper we will establish new results including companion inequalities of (5) and (6) and the conditions for such inequalities to become equalities. Especially we will present new results in the non-singular Hermitian matrix case in which no positive definite matrices are required. The new results in this case are related but different, because they hold for different extra conditions. No such results even involving the Hadamard product, corresponding to (3) and (4) in the positive definite matrix case, have been reported previously. In Section 2, we provide basic results. In Sections 3 and 4, we discuss the non-negative definite and non-singular Hermitian cases, respectively. In Section 5, we establish Kantorovich-type inequalities in the positive definite matrix case. Concluding remarks are made in Section 6.

2. Basic lemmas

Lemma 1. *For compatibly partitioned matrices A, B, C and D, we have*

$$(A \bowtie C)(B \bowtie D) = AB \bowtie CD, \tag{7}$$
$$(A \bowtie B)^H = A^H \bowtie B^H, \tag{8}$$

$$(A \bowtie B)^+ = A^+ \bowtie B^+, \tag{9}$$
$$A \bowtie B \geqslant 0 (> 0) \text{ if } A, B \geqslant 0 (> 0), \tag{10}$$
$$A * B = Z^H (A \bowtie B) Z, \tag{11}$$
$$(A \bowtie B)(\alpha) = A * B, \tag{12}$$

where Z is a selection matrix with elements 0 or 1 such that $Z^H Z = I$, (11) holds if A and B are both square, and α in (12) indicates the rows and columns selected by Z in (11).

Proof. These all follow from the properties of the Kronecker product. □

Note that the partition of Z depends only on the partitions of A and of B. Throughout this paper, Z is determined by (11) and α is characterised by (12) corresponding to Z.

Lemma 2

$$X^H VX(X^H X)^+ X^H VX \leqslant X^H V^2 X, \tag{13}$$
$$X^H VX(X^H V^2 X)^+ X^H VX \leqslant X^H X, \tag{14}$$

where $V \in H(n)$. Equality holds in (13) if and only if $VX = X^0 VX$, and equality holds in (14) if and only if $X = (VX)^0 X$.

Proof. Use $X(X^H X)^T X^H \leqslant I$ and $VX(X^H V^2 X)^+ X^H V \leqslant I$. □

Lemma 3

$$X^H V^2 X \leqslant a X^H VX(X^H X)^+ X^H VX, \tag{15}$$
$$X^H V^2 X - X^H VX(X^H X)^+ X^H VX \leqslant b X^H VX, \tag{16}$$
$$X^H X \leqslant c X^H VX(X^H V^2 X)^+ X^H VX, \tag{17}$$
$$X^H X - X^H VX(X^H V^2 X)^+ X^H VX \leqslant d X^H VX, \tag{18}$$

where $V \geqslant 0$, $X = V^0 X$, $a = c = (\lambda + \mu)^2/(4\lambda\mu)$, $b = \left(\sqrt{\lambda} - \sqrt{\mu}\right)^2$, $d = \left(\sqrt{\lambda} - \sqrt{\mu}\right)^2 \Big/ (\lambda\mu)$, and λ and μ are the largest and smallest eigenvalues of V, respectively. Equality holds in (15) if and only if

$$X^H X = \frac{\lambda + \mu}{2} X^H V^2 X \quad \text{and} \quad X^H V^2 X = \frac{\lambda + \mu}{2\lambda\mu} X^H VX, \quad \text{or} \quad X = 0;$$

in (16) if and only if

$$X^H V^2 X = (\lambda + \mu - \sqrt{\lambda\mu}) X^H VX \text{ and } X^H X = \tfrac{1}{\sqrt{\lambda\mu}} X^H VX,$$
$$\text{or} \quad X = 0, \quad \text{or} \quad \lambda = \mu;$$

in (17) if and only if

$$X^H V^2 X = \frac{\lambda + \mu}{2} X^H VX \quad \text{and} \quad X^H X = \frac{\lambda + \mu}{2\lambda\mu} X^H VX, \quad \text{or} \quad X = 0;$$

and in (18) if and only if

$$X^H V^2 X = \sqrt{\lambda\mu} X^H VX \quad \text{and} \quad X^H X = \tfrac{\lambda + \mu - \sqrt{\lambda\mu}}{\lambda\mu} X' VX,$$
$$\text{or} \quad X = 0, \quad \text{or} \quad \lambda = \mu.$$

Proof. See [7]. See also [12]. □

Definition of the Schur Complement (*see e.g.* [11,16,2, pp. 20 and 21]). Let $\beta \subset \langle n \rangle$ with $\beta' = \langle n \rangle \setminus \beta$ (i.e. β and β' are a partition of $\langle n \rangle$ here), let $M \in C^{n \times n}$ be invertible and let the inverse of $M(\beta')$ be denoted as $M(\beta')^{-1}$. For

$$U^H M U = \begin{pmatrix} M(\beta) & M(\beta,\beta') \\ M(\beta',\beta) & M(\beta') \end{pmatrix} \in C^{n\times n}, \tag{19}$$

where $U \in C^{n\times n}$ is a permutation matrix, the Schur complement of $M(\beta')$, also denoted as M/β', is

$$U^H M U/M(\beta') = M(\beta) - M(\beta,\beta')M(\beta')^{-1}M(\beta',\beta). \quad \square \tag{20}$$

Lemma 4. *Let $Q \in C^{n\times n}$ be arbitrary, $M \in H(n)$ be invertible, $\beta \subset \langle n\rangle$ and $\beta' = \langle n\rangle \setminus \beta$. If $M^{-1}(\beta') > 0$, then $M(\beta)$ is invertible and*

$$Q(\beta)^H M(\beta)^{-1} Q(\beta) \leqslant (Q^H M^{-1} Q)(\beta), \tag{21}$$

and (21) becomes equality if and only if

$$M^{-1}(\beta',\beta)Q(\beta) = -M^{-1}(\beta')Q(\beta',\beta). \tag{22}$$

Proof. By $M \in H(n)$ and the structure of U given in (19), we have

$$U^H M^{-1} U = \begin{pmatrix} M^{-1}(\beta) & M^{-1}(\beta,\beta') \\ M^{-1}(\beta',\beta) & M^{-1}(\beta') \end{pmatrix} \in H(n), \tag{23}$$

$$U^H Q U = \begin{pmatrix} Q(\beta) & Q(\beta,\beta') \\ Q(\beta',\beta) & Q(\beta') \end{pmatrix}, \tag{24}$$

$$U^H Q^H M^{-1} Q U = \begin{pmatrix} (Q^H M^{-1} Q)(\beta) & (Q^H M^{-1} Q)(\beta,\beta') \\ (Q^H M^{-1} Q)(\beta',\beta) & (Q^H M^{-1} Q)(\beta') \end{pmatrix} \in H(n). \tag{25}$$

As M^{-1} and $M^{-1}(\beta')$ are invertible, by ([2, Theorem 1.2]) we see that M^{-1}/β' and $M(\beta)$ are both invertible and

$$M^{-1}/\beta' = M(\beta)^{-1}. \tag{26}$$

Using (23), (24) and

$$W = \begin{pmatrix} I & 0 \\ -(M^{-1}(\beta'))^{-1}M^{-1}(\beta',\beta) & I \end{pmatrix},$$

$$W^{-1} = \begin{pmatrix} I & 0 \\ (M^{-1}(\beta'))^{-1}M^{-1}(\beta',\beta) & I \end{pmatrix},$$

we get

$$W^H U^H M^{-1} U W = \mathrm{diag}(M^{-1}/\beta', M^{-1}(\beta')), \tag{27}$$

$$W^{-1} U^H Q U = \begin{pmatrix} Q(\beta) & Q(\beta,\beta') \\ X & Y \end{pmatrix}, \tag{28}$$

$$X = (M^{-1}(\beta'))^{-1}M^{-1}(\beta',\beta)Q(\beta) + Q(\beta',\beta),$$
$$Y = (M^{-1}(\beta'))^{-1}M^{-1}(\beta',\beta)Q(\beta,\beta') + Q(\beta').$$

Using $M^{-1}(\beta') > 0$, (25–28) and

$$U^H Q^H M^{-1} Q U = (W^{-1}U^H Q U)^H (W^H U^H M^{-1} U W)(W^{-1} U^H Q U), \tag{29}$$

we get

$$\begin{aligned} (Q^H M^{-1} Q)(\beta) &= (Q(\beta)^H, X^H)\mathrm{diag}(M(\beta)^{-1}, M^{-1}(\beta'))(Q(\beta)^H, X^H)^H \\ &= Q(\beta)^H M(\beta)^{-1} Q(\beta) + X^H M^{-1}(\beta') X \\ &\geqslant Q(\beta)^H M(\beta)^{-1} Q(\beta). \end{aligned}$$

This means (21) holds, and equality holds if and only if $X^H M^{-1}(\beta')X = 0$, i.e. $M^{-1}(\beta')X = M^{-1}(\beta',\beta)Q(\beta) + M^{-1}(\beta')Q(\beta',\beta) = 0$, i.e. (22) holds. $\square$

Lemma 5. *Consider* $F, G, T \in C^{n\times n}$. *If* $F, G > 0$, *then* $F \geqslant TG^{-1}T^H \Longleftrightarrow G \geqslant T^H F^{-1}T$, *and further* $F = TG^{-1}T^H \Longleftrightarrow G = T^H F^{-1}T$.

Proof. Consider

$$E = \begin{pmatrix} F & T \\ T^H & G \end{pmatrix}, \quad P = \begin{pmatrix} I & 0 \\ -G^{-1}T^H & I \end{pmatrix}, \quad S = \begin{pmatrix} I & -F^{-1}T \\ 0 & I \end{pmatrix},$$

so that

$$P^H EP = \mathrm{diag}(F - TG^{-1}T^H, G),$$
$$S^H ES = \mathrm{diag}(F,\ G - T^H F^{-1}T).$$

It follows that $F \geqslant TG^{-1}T^H \Longleftrightarrow P^H EP \geqslant 0 \Longleftrightarrow S^H ES \geqslant 0 \Longleftrightarrow G \geqslant T^H F^{-1}T$. Further $F = TG^{-1}T^H \Longleftrightarrow \mathrm{rank}(P^H EP) = \mathrm{rank}G = n \Longleftrightarrow \mathrm{rank}(S^H ES) = \mathrm{rank}F = n \Longleftrightarrow G = T^H F^{-1}T$. $\square$

3. Non-negative definite matrix case

Theorem 1. *Consider* $A \geqslant 0$ *and* $B \geqslant 0$ *partitioned consistently, and* Z *as defined in* (11). *Then* (6) *holds, and* (6) *becomes equality if and only if*

$$(A^+ \bowtie B^0 + A^0 \bowtie B^+)Z = (A^0 \bowtie B^0)Z(A * B)^+(A * B^0 + A^0 * B).$$

Furthermore,

$$(A * B^0 + A^0 * B)(A * B^+ + A^+ * B + 2A^0 * B^0)^+(A * B^0 + A^0 * B) \leqslant A * B, \tag{30}$$

and (30) *becomes equality if and only if*

$$(A \bowtie B)Z = (A^0 \bowtie B + A \bowtie B^0)Z(A * B^+ + A^+ * B + 2A^0 * B^0)^+(A * B^0 + A^0 * B).$$

Proof. We establish Theorem 1 by using Lemma 2, $X = (A^{1/2} \bowtie B^{1/2})Z$ and $V = I \bowtie B^+ + A^+ \bowtie I \geqslant 0$ with

$$X^H X = A * B,$$
$$VX = (A^{1/2} \bowtie (B^+)^{1/2} + (A^+)^{1/2} \bowtie B^{1/2})Z,$$
$$X^H VX = A * B^0 + A^0 * B,$$
$$X^H V^2 X = A * B^+ + A^+ * B + 2A^0 * B^0. \quad \square$$

As a special case of (11) and (12), we have (see e.g. [14])

$$A \odot B = P^H(A \otimes B)P = (A \otimes B)(\alpha), \tag{31}$$

where $A, B \in C^{n\times n}$, P is a selection matrix and $\alpha \subset \langle n^2 \rangle$.

Corollary 1. *Consider* $A, B \geqslant 0$. *Then* (5) *holds, and* (5) *becomes equality if and only if*

$$(A^+ \otimes B^0 + A^0 \otimes B^+)P = (A^0 \otimes B^0)P(A \odot B)^+(A \odot B^0 + A^0 \odot B).$$

Also

$$(A \odot B^0 + A^0 \odot B)(A \odot B^+ + A^+ \odot B + 2A^0 \odot B^0)^+ \times (A \odot B^0 + A^0 \odot B) \leqslant A \odot B, \tag{32}$$

while (32) *becomes equality if and only if*

$$(A \otimes B)P = (A^0 \otimes B + A \otimes B^0)P(A \odot B^+ + A^+ \odot B + 2A^0 \odot B^0)^+(A \odot B^0 + A^0 \odot B),$$

where P *is determined by* (31).

Clearly (30) is a companion of (6), and specially, (32) is a companion of (5).

4. Non-singular Hermitian matrix case

Theorem 2. *Consider non-singular square matrices* $A \in H(m)$ *and* $B \in H(p)$ *partitioned consistently,* $\alpha \subset \langle mp \rangle$ *determined by* (11) *and* (12) *via* Z*, and* $\alpha' = \langle mp \rangle \setminus \alpha$*. If* $(A \bowtie B)^{-1}(\alpha') > 0$*, then* $A * B$ *is invertible and*

$$(A * I + B * I)(A * B)^{-1}(A * I + B * I) \leqslant A * B^{-1} + A^{-1} * B + 2I, \tag{33}$$

and inequality becomes equality if and only if

$$(A \bowtie B)^{-1}(\alpha', \alpha)(A * I + B * I) = -(A \bowtie B)^{-1}(\alpha')(A \bowtie I + I \bowtie B)(\alpha', \alpha). \tag{34}$$

Proof. Let $Q = A \bowtie I + I \bowtie B, M = A \bowtie B$. By (8) and (9), we see $Q \in H(mp)$, and $M \in H(mp)$ is invertible. Note that $M^{-1}(\alpha') = (A \bowtie B)^{-1}(\alpha') > 0$, i.e. Q and M are in accordance with Lemma 4. So

$$\begin{aligned}
(Q^H M^{-1} Q)(\alpha) &= ((A \bowtie I + I \bowtie B)(A^{-1} \bowtie B^{-1})(A \bowtie I + I \bowtie B))(\alpha) \\
&= (A \bowtie B^{-1} + A^{-1} \bowtie B + 2I \bowtie I)(\alpha) \\
&= (A \bowtie B^{-1})(\alpha) + (A^{-1} \bowtie B)(\alpha) + 2(I \bowtie I)(\alpha) \\
&= A * B^{-1} + A^{-1} * B + 2I * I, \\
Q(\alpha) = Q(\alpha)^H &= (A \bowtie I + I \bowtie B)(\alpha) = A * I + I * B, \\
M(\alpha)^{-1} &= (A \bowtie B)(\alpha)^{-1} = (A * B)^{-1}.
\end{aligned}$$

By Lemma 4 we see that (33) holds:

$$\begin{aligned}
&(A * I + B * I)(A * B)^{-1}(A * I + B * I) \\
&\quad = Q(\alpha)^H (A \bowtie B)(\alpha)^{-1} Q(\alpha) \leqslant (Q^H (A \bowtie B)^{-1} Q)(\alpha) \\
&\quad = A * B^{-1} + A^{-1} * B + 2I * I.
\end{aligned}$$

Equality in (33) holds if and only if (34) is valid, since

$$\begin{aligned}
&M^{-1}(\alpha', \alpha)Q(\alpha) + M^{-1}(\alpha')Q(\alpha', \alpha) \\
&\quad = (A \bowtie B)^{-1}(\alpha', \alpha)Q(\alpha) + (A \bowtie B)^{-1}(\alpha')Q(\alpha', \alpha) \\
&\quad = (A \bowtie B)^{-1}(\alpha', \alpha)(A * I + I * B) \\
&\qquad + (A \bowtie B)^{-1}(\alpha')(A \bowtie I + I \bowtie B)(\alpha', \alpha) = 0. \qquad \square
\end{aligned}$$

Corollary 2. *Let* $A \in H(n)$ *and* $B \in H(n)$ *be invertible,* α *be determined by* (31) *and* $\alpha' = \langle n^2 \rangle \setminus \alpha$*. If* $(A \otimes B)^{-1}(\alpha') > 0$*, then* $A \odot B$ *is invertible and inequality* (4) *holds, and* (4) *becomes equality if and only if*

$$(A \otimes B)^{-1}(\alpha', \alpha)(A \odot I + B \odot I) = -(A \otimes B)^{-1}(\alpha')(A \otimes I + I \otimes B)(\alpha', \alpha). \tag{35}$$

By Corollary 2 we can easily find equality conditions for (1) and (2). Choose $A \in H(2)$ and $B \in H(2)$ to be not non-negative definite and to be non-singular:

$$A = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \quad B = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix}.$$

We see that (4) still holds. This is because the condition of Lemma 4 is met, as $\alpha = \{1, 4\}, \alpha' = \{2, 3\}$ and

$$(A \otimes B)^{-1}(\alpha') = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} > 0.$$

Also, we see that (3) does not hold, and accordingly is not equivalent to (4) when A and B are not assumed to be positive definite. We need a Theorem to extend (3). This is also done by using Lemma 4, just as Theorem 2 extends (4).

Theorem 3. *Consider non-singular square matrices* $A \in H(m)$ *and* $B \in H(p)$ *partitioned consistently,* $\alpha \subset \langle mp \rangle$ *determined by* (11) *and* (12)*, and* $\alpha' = \langle mp \rangle \setminus \alpha$*. If* $A \bowtie I + I \bowtie B$ *is non-singular and* $(A \bowtie B^{-1} + A^{-1} \bowtie B + 2I)^{-1}(\alpha') > 0$*, then*

$$(A * I + B * I)(A * B^{-1} + A^{-1} * B + 2I)^{-1}(A * I + B * I) \leqslant A * B, \tag{36}$$

and inequality becomes equality if and only if

$$\begin{aligned}&(A \bowtie B^{-1} + A^{-1} \bowtie B + 2I)^{-1}(\alpha', \alpha)(A * I + I * B)\\ &\quad = -(A \bowtie B^{-1} + A^{-1} \bowtie B + 2I)^{-1}(\alpha')(A * I + I * B)(\alpha', \alpha).\end{aligned} \tag{37}$$

Proof. Let $Q = Q^H = A \bowtie I + I \bowtie B$ and $M = Q^H(A \bowtie B)^{-1}Q$. Clearly $Q \in H(mp)$ is non-singular. By (9), $M \in H(mp)$ is non-singular and

$$\begin{aligned}M &= Q^H(A \bowtie B)^{-1}Q\\ &= (A \bowtie I + I \bowtie B)(A^{-1} \bowtie B^{-1})(A \bowtie I + I \bowtie B)\\ &= A \bowtie B^{-1} + A^{-1} \bowtie B + 2I,\\ M^{-1}(\alpha') &= (A \bowtie B^{-1} + A^{-1} \bowtie B + 2I)^{-1}(\alpha') > 0,\end{aligned}$$

i.e., Q and M meet the conditions of Lemma 4. Further, by (12),

$$\begin{aligned}&Q(\alpha) = Q(\alpha)^H = (A \bowtie I + I \bowtie B)(\alpha) = A * I + I * B,\\ &M(\alpha)^{-1} = (A \bowtie B^{-1} + A^{-1} \bowtie B + 2I)(\alpha')^{-1}\\ &\qquad\quad = (A * B^{-1} + A^{-1} * B + 2I)^{-1},\\ &(Q^H M^{-1} Q)(\alpha) = (Q^H(Q^H(A \bowtie B)^{-1}Q)^{-1}Q)(\alpha)\\ &\qquad\qquad\qquad = (A \bowtie B)(\alpha) = A * B.\end{aligned}$$

By Lemma 4,

$$\begin{aligned}&(A * I + B * I)(A * B^{-1} + A^{-1} * B + 2I)^{-1}(A * I + B * I)\\ &\quad = Q(\alpha)^H M(\alpha)^{-1} Q(\alpha) \leqslant Q^H M^{-1} Q(\alpha) = A * B,\end{aligned}$$

i.e., (36) holds, and (36) becomes equality if and only if

$$\begin{aligned}&M^{-1}(\alpha', \alpha)Q(\alpha)\\ &\quad = -M^{-1}(\alpha')Q(\alpha', \alpha)\\ &\quad = (A \bowtie B^{-1} + A^{-1} \bowtie B + 2I)^{-1}(\alpha', \alpha)(A * I + I * B)\\ &\quad = -(A * B^{-1} + A^{-1} * B + 2I)^{-1}(\alpha')(A * I + I * B)(\alpha', \alpha). \qquad \square\end{aligned}$$

Note that it is necessary to assume $A \bowtie I + I \bowtie B$ to be non-singular, as it can be singular for some non-singular matrices A and B, e.g. $A = -B = I$. If we assume $A > 0$ and $B > 0$, we can easily see that $A \bowtie I + I \bowtie B > 0$ is non-singular, and $(A \bowtie B^{-1} + A^{-1} \bowtie B + 2I)^{-1}(\alpha') > 0$.

Corollary 3. *Let $A, B \in H(n)$ be non-singular, α be specified by (31) and $\alpha' = \langle n^2 \rangle - \alpha$. If $A \otimes I + I \otimes B$ is non-singular and $(A \otimes B^{-1} + A^{-1} \otimes B + 2I)^{-1}(\alpha') > 0$, then $A \odot B$ is non-singular and (3) holds, and (3) is equality if and only if (38) holds:*

$$\begin{aligned}&(A \otimes B^{-1} + A^{-1} \otimes B + 2I)^{-1}(\alpha', \alpha)(A \odot I + B \odot I)\\ &\quad = -(A \otimes B^{-1} + A^{-1} \otimes B + 2I)^{-1}(\alpha')(A \otimes I + I \otimes B)(\alpha', \alpha).\end{aligned} \tag{38}$$

5. Positive definite matrix case with counterpart inequalities

Theorem 4. *Consider $A > 0$ and $B > 0$ partitioned consistently. Then (33) and (36) hold, and inequality in (33) becomes equality $\Longleftrightarrow$ inequality in (36) becomes equality $\Longleftrightarrow$ (34) holds $\Longleftrightarrow$ (37) holds.*

Proof. By (10), $F = A^{-1} + A * B^{-1} + 2I$ and $G = A * B$ are positive definite. By setting $T = A * I + B * I$, we see that (33) holds if and only if $F \geqslant TG^{-1}T^H$, and (36) holds if and only if $G \geqslant T^H F^{-1} T$. Further,

(33) and (36) hold, by Corollaries 2 and 3. We see from Lemma 5, Theorems 2 and 3 that equalities hold *equivalently* in (33), (34), (36) and (37). □

Corollary 4. *Let* $A, B > 0$. *Then* (3) *and* (4) *hold, and equalities hold equivalently in* (3), (4), (35) *and* (38).

By Corollary 4 we can easily get (1) and (2) and their conditions to become equality.

We now give four counterpart inequalities for Theorem 1.

Theorem 5. *Consider* $A > 0$ *and* $B > 0$ *partitioned consistently. Then*

$$A * B^{-1} + A^{-1} * B + 2I \leqslant a(A * I + I * B)(A * B)^{-1}(A * I + I * B), \tag{39}$$

$$A * B^{-1} + A^{-1} * B + 2I - (A * I + I * B)(A * B)^{-1}(A * I + I * B) \leqslant b(A * I + I * B), \tag{40}$$

$$A * B \leqslant c(A * I + I * B)(A * B^{-1} + A^{-1} * B + 2I)^{-1}(A * I + I * B), \tag{41}$$

$$A * B - (A * I + I * B)(A * B^{-1} + A^{-1} * B + 2I)^{-1}(A * I + I * B) \leqslant d(A * I + I * B), \tag{42}$$

where a, b, c *and* d *are as expressed in Lemma* 3, *with* λ *and* μ *being the largest and smallest eigenvalues of* $I \bowtie B^{-1} + A^{-1} \bowtie I > 0$.

Proof. Use Lemma 3, $X = (A^{1/2} \bowtie B^{1/2})Z$ and $V = I \bowtie B^{-1} + A^{-1} \bowtie I > 0$ with

$$X^H X = A * B,$$
$$VX = (A^{1/2} \bowtie B^{-1/2} + A^{-1/2} \bowtie B^{1/2})Z,$$
$$X^H V X = A * I + I * B,$$
$$X^H V^2 X = A * B^{-1} + A^{-1} * B + 2I,$$

where Z is defined as in (11). □

6. Concluding remarks

We have presented several inequalities involving the Khatri–Rao product, with their equality conditions. As the Khatri–Rao product is a generalised Hadamard product, the theorems reduce to those results involving the Hadamard product as a special case, including (1)–(5). Theorem 1 in the non-negative definite case actually gives Cauchy–Schwarz-type inequalities. Theorem 5 in the positive definite case gives Kantorovich-type inequalities, as a counterpart result of Theorem 1. Theorems 2 and 3 are established for the non-singular Hermitian case based on the Schur complement method. They are valid for different conditions, but become the equivalent inequalities for the positive definite case, namely Theorem 4.

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